A Globally Exponentially Convergent Immersion and Invariance Speed Observer for Mechanical Systems with Non–Holonomic Constraints

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Problem Formulation

- Consider general n-dof mechanical systems with non–holonomic constraints

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla U(q) &= G(q)u + Z(q)\lambda, \\ Z^\top(q)\dot{q} &= 0, \end{aligned}$$

  where $q, \dot{q} \in \mathbb{R}^n$ generalized positions and velocities, $u \in \mathbb{R}^m$ control input, $Z(q)\lambda$ constraint forces with $Z : I\!R^n \to I\!R^{n\times k}$, $\lambda \in \mathbb{R}^k$, $G : I\!R^n \to I\!R^{n\times m}$ input matrix, $M : I\!R^n \to I\!R^{n\times n}$, $M = M^\top > 0$ mass matrix, and $U : I\!R^n \to I\!R$ potential energy function. $C(q,\dot{q})\dot{q}$ vector of Coriolis and centrifugal forces, defined via the Christoffel symbols.

- Assumptions
  - q measurable and
  - the system is forward complete.

- Objective

  Design a globally asymptotically convergent observer for $\dot{q}$.

Observer Design: The I&I Approach

- The dynamical system

$$\dot{\eta} = \alpha(q, \eta),$$

with $\eta \in \mathbb{R}^n$, is called an I&I observer for the mechanical system if there exists a full rank matrix $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and a vector function $\beta : \mathbb{R}^n \to \mathbb{R}^n$, such that the manifold

$$\mathcal{M} := \{(\eta, q, \dot{q}) : \beta(q) = \eta + \mathcal{T}^\top(q)\dot{q}\} \subset \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$$

is positively invariant and attractive.

- The asymptotic estimate of $\dot{q}$ is given by

$$\mathcal{T}^{-\top}(\beta - \eta).$$

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Main Result

Proposition There exist smooth mappings $A : \mathbb{R}^{3n-2k+1} \times \mathbb{R}^{n} \times \mathbb{R}^{m} \to \mathbb{R}^{3n-2k+1}$, $B : \mathbb{R}^{n} \to \mathbb{R}^{(n-k)\times(3n-2k+1)}$ and $\mathcal{N} : \mathbb{R}^{n} \to \mathbb{R}^{n-k} \times \mathbb{R}^{n}$, with the latter left invertible, such that the dynamical system

$$\dot{\chi} = A(\chi, q, u) \qquad (O1)$$

with state $\chi \in \mathbb{R}^{3n-2k+1}$, inputs q and u, and output

$$\eta = B(q)\chi, \qquad (O2)$$

has the following property

$$\lim_{t\to\infty} e^{\alpha t}[\mathcal{N}(q)\dot{q}(t) - \eta(t)] = 0,$$

for some $\alpha > 0$ and for all initial conditions $(q(0), \dot{q}(0), \chi(0)) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{3n-2k+1}$. That is, (O1), (O2) is a globally exponentially convergent speed observer for the mechanical system.

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A Suitable Hamiltonian Representation

- The system can be written in port-Hamiltonian form as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{pmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u + \begin{bmatrix} 0 \\ Z(q) \end{bmatrix} \lambda,$$
$$Z^\top(q) \nabla_p H(q,p) = 0,$$

where $p = M(q)\dot{q}$, and

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + U(q).$$

- Restricted to the constrained space, it can be represented as (van der Schaft'2000)

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{S}(q) \\ -\tilde{S}^\top(q) & J(q,\tilde{p}) \end{bmatrix} \begin{pmatrix} \nabla_q H_c(q,\tilde{p}) \\ \nabla_{\tilde{p}} H_c(q,\tilde{p}) \end{pmatrix} + \begin{bmatrix} 0 \\ G_c(q) \end{bmatrix} u,$$

with

$$H_c(q,\tilde{p}) = \frac{1}{2} \tilde{p}^\top \tilde{M}^{-1}(q) \tilde{p} + U(q),$$

cont'd

where $\tilde{M} : I\!R^n \to I\!R^{(n-k)\times(n-k)}$ is defined as

$$\tilde{M}^{-1}(q) = m_{11}(q) - m_{12}(q) m_{22}^{-1}(q) m_{12}(q).$$

with m_{ij} the partition of M^{-1} (induced by the constraints),

$$\tilde{p} = \tilde{S}^\top(q) p,$$

$\tilde{S} : I\!R^n \to \mathbb{R}^{n\times(n-k)}$ is a full–rank right annihilator of $Z^\top$ and $G_c : I\!R^n \to \mathbb{R}^{(n-k)\times m}$ is the constrained input matrix G. The (ij)-th element of $J : I\!R^n \times I\!R^{n-k} \to I\!R^{(n-k)\times(n-k)}$ is given by

$$J_{ij}(q, \tilde{p}) = -p^\top [\tilde{S}_i, \tilde{S}_j], \tag{1}$$

where $\tilde{S}_i$ is the i-th column of $\tilde{S}$, and $[\tilde{S}_i, \tilde{S}_j]$ is the standard Lie bracket. Recalling that

$$[\tilde{S}_i, \tilde{S}_j] = -[\tilde{S}_j, \tilde{S}_i]$$

we conclude that J is skew–symmetric.

Some Definitions

- Introduce a factorization, (e.g., Cholesky) $\tilde{M}^{-1}(q) = T^\top(q)T(q)$, where $T : I\!R^n \to I\!R^{(n-k)\times(n-k)}$ is a full rank matrix.
- Define
  $L : I\!R^n \to I\!R^{n\times(n-k)}$, $F : I\!R^n \times I\!R^m \to I\!R^{n-k}$, $S : I\!R^n \times I\!R^{n-k} \to I\!R^{(n-k)\times(n-k)}$

$$\begin{aligned} L(q) &= \tilde{S}(q)T^\top(q), \\ F(q,u) &= T(q)[G_c(q)u - \tilde{S}^\top(q)\nabla U(q)] \\ S(y,x) &= TJT^\top + \sum_{i=1}^{n}[((\nabla_{q_i}T)T^{-1}x)(L^\top e_i)^\top - (L^\top e_i)((\nabla_{q_i}T)T^{-1}x)^\top]. \end{aligned}$$

  where

$$(y,x) = (q, T(q)\tilde{p}),$$

  and e_i the i–th basis vector of $\mathbb{R}^{n-k}$.
- Notice that, since q and u are measurable, L and F are known. Moreover, L is a left-invertible matrix.

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A Key Lemma

- The system admits a state space representation of the form

$$\begin{aligned} \dot{y} &= L(y)x \\ \dot{x} &= S(y,x)x + F(y,u), \qquad (\heartsuit) \end{aligned}$$

  and S verifies the following properties.

  (i) S is skew–symmetric.

  (ii) S is linear in the second argument.

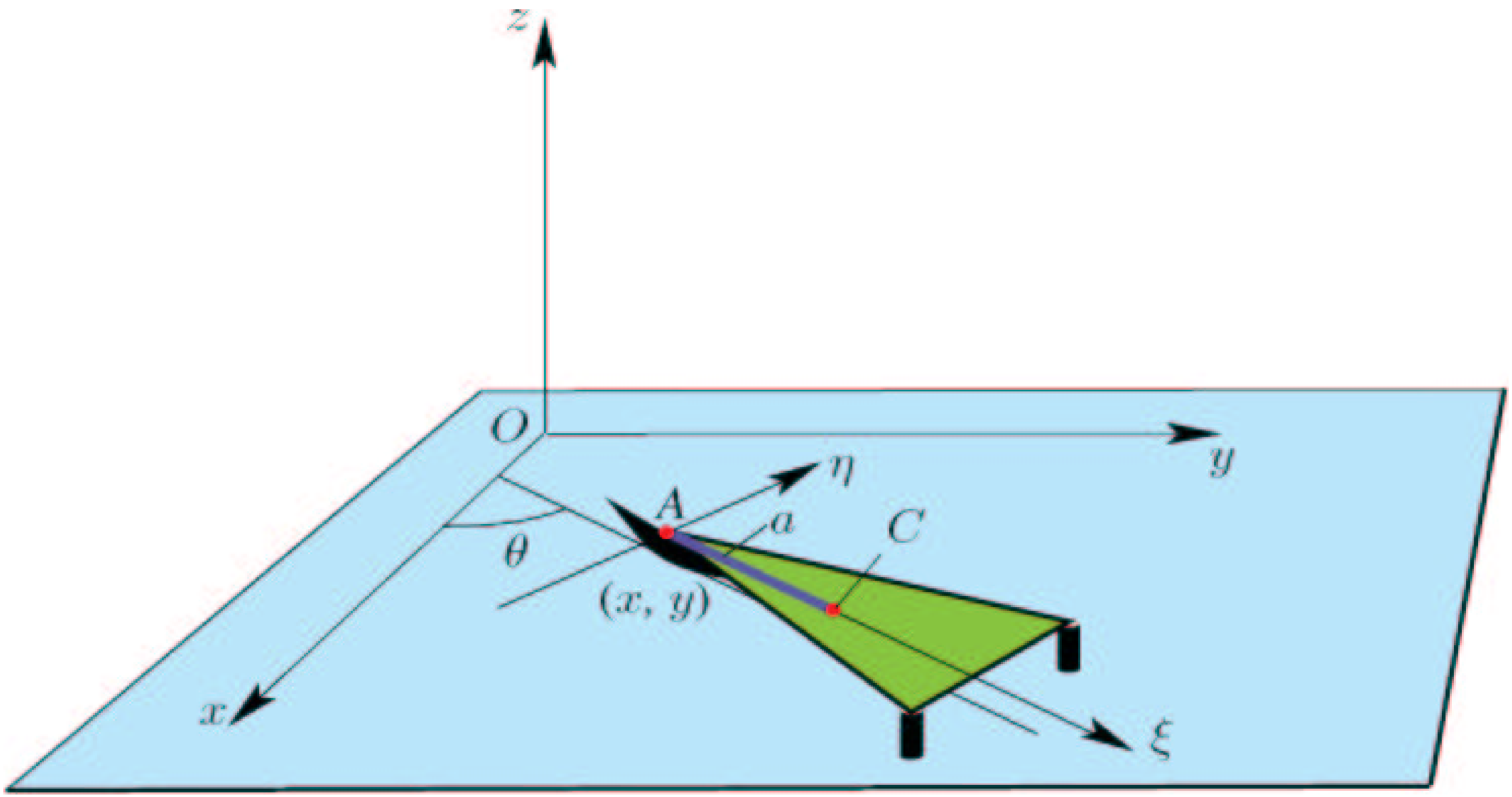
  (iii) There exists a mapping $\bar{S} : \mathbb{R}^n \times \mathbb{R}^{n-k} \to \mathbb{R}^{(n-k)\times(n-k)}$ such that

$$S(y,x)\bar{x} = \bar{S}(y,\bar{x})x,$$

  for all $y \in \mathbb{R}^n$, $x, \bar{x} \in \mathbb{R}^{n-k}$.

- The lemma implies that the speed observer problem can be recast as an observer problem for ($\heartsuit$) with output y.

Example 1: The Chaplygin Sleigh



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- Model

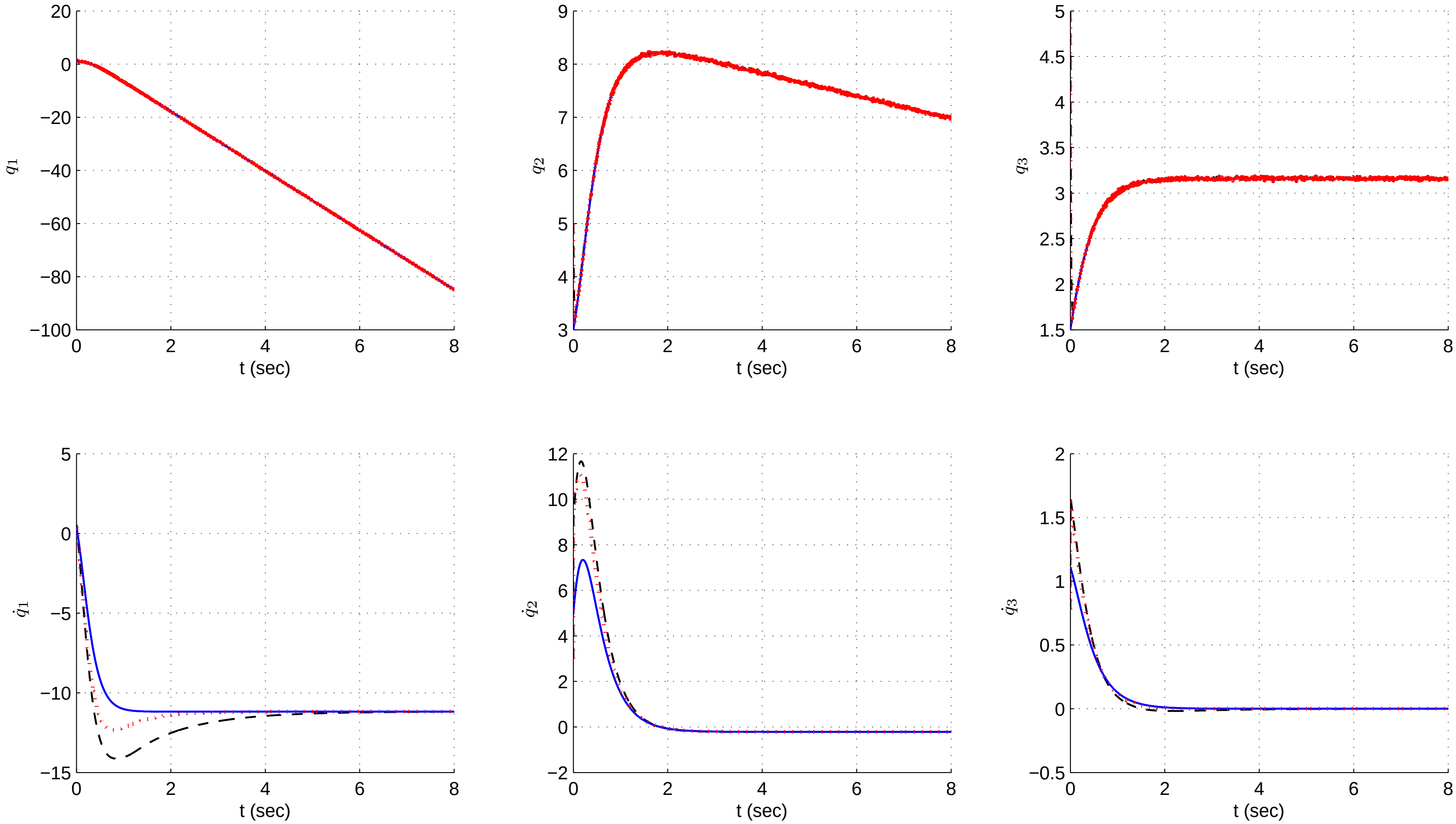
$$M(q) = \begin{bmatrix} m & 0 & -ma\sin(q_3) \\ 0 & m & ma\cos(q_3) \\ -ma\sin(q_3) & ma\cos(q_3) & I + ma^2 \end{bmatrix}, \quad Z(q) = \begin{bmatrix} -\sin(q_3) \\ \cos(q_3) \\ 0 \end{bmatrix},$$

- m is the mass of the rigid body, I is the moment of inertia of the rigid body about its center of mass and a denotes the fixed distance between the knife edge and the center of mass. The body is moving on the ground, that is, $U(q) = 0$ and $u = 0$.
- Simulation parameters and IC's

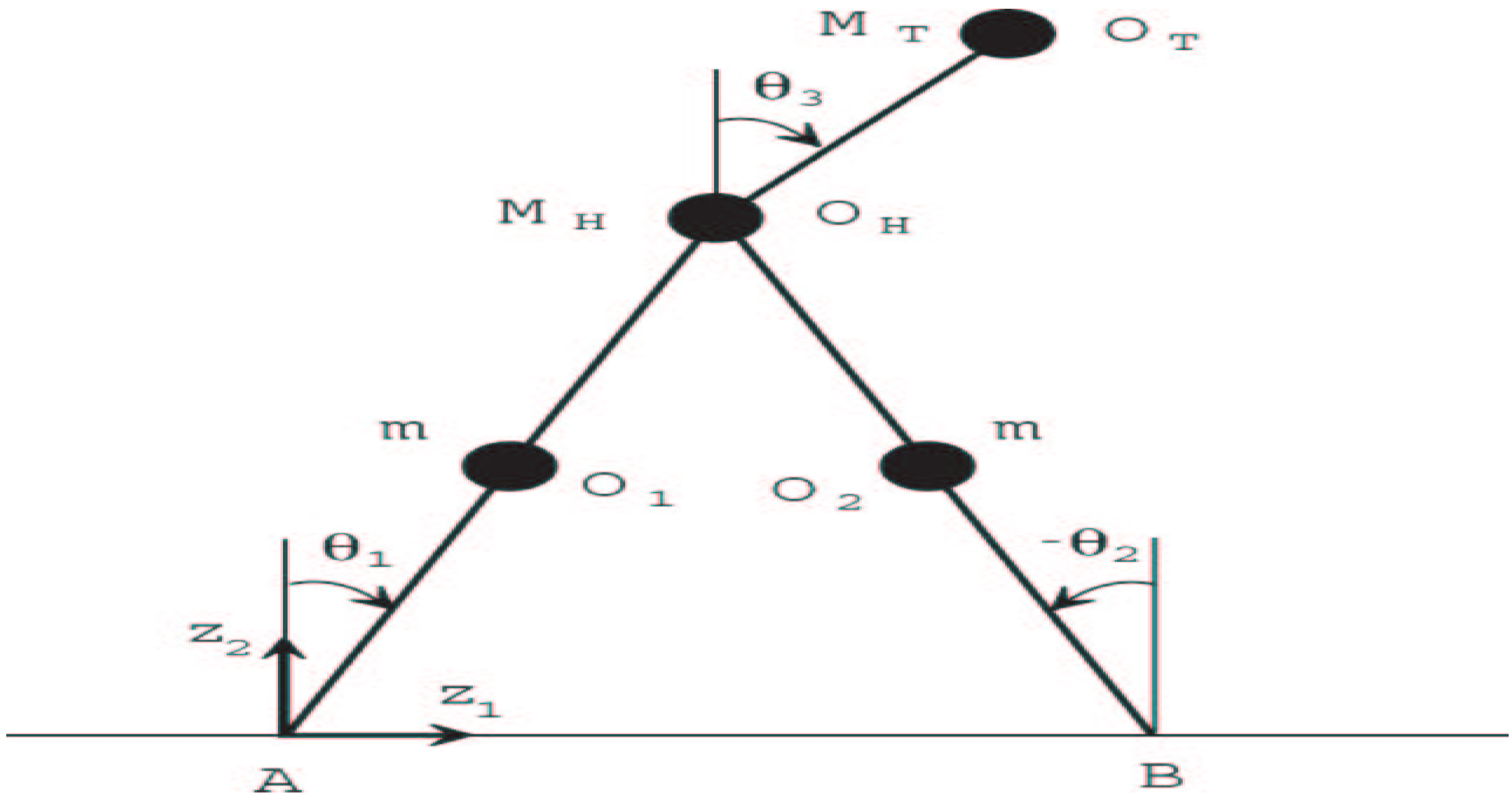
$m = 1$	$y(0) = (1, 3, 1.5)$
$a = 2$	$x(0) = (5, 10)$
$I = 5$	$\hat{y}(0) = (2, 5, 5)$
$k_1 = 0.8, 1.7$	$\hat{x}(0) = (3, 7)$
$k_2 = 4$	$\xi_1(0) = (2, 2)$
$k_3 = 5$	$r(0) = 3$

cont'd

q, $\hat{y}$ (top row) and of $\dot{q}$, $\hat{\dot{q}}$ (bottom row), for $k_1 = 0.8$ (dashed lines) and $k_1 = 1.7$ (dotted lines).



Example 2: A Walking Robot



Conclusions

- A definite affirmative answer has been given to the question of existence of a globally convergent speed observer for general mechanical systems with non–holonomic constraints.
- No assumption is made on the existence of an upperbound for the inertia matrix, hence the result is applicable for robots with prismatic joints.
- The only requirement is that the system is forward complete, *i.e.*, that trajectories of the system exist for all times $t \geq 0$—which is a rather weak condition.
- In some sense, our contribution should be interpreted more as an existence result than an actual, practically implementable, algorithm. Leaving aside the high complexity of the observer dynamics, the difficulty stems from the fact that a key function is defined via an integral expression, whose explicit analytic solution cannot be guaranteed *a priori*.